

Configuration categories and homotopy automorphisms

Michael S. Weiss

ABSTRACT. Let M be a smooth compact manifold with boundary. Under some geometric conditions on M , a homotopical model for the pair $(M, \partial M)$ can be recovered from the configuration category of $M \setminus \partial M$. The grouplike monoid of derived homotopy automorphisms of the configuration category of $M \setminus \partial M$ then acts on the homotopical model of $(M, \partial M)$. That action is compatible with a better known homotopical action of the homeomorphism group of $M \setminus \partial M$ on $(M, \partial M)$.

1. Introduction

The term *configuration category* of a topological manifold M has a number of interpretations [3]. In one of them, which is compelling because it makes a direct connection with configuration spaces, it is a category enriched in topological spaces such that the object space is

$$\coprod_{k \geq 0} \text{emb}(\underline{k}, M)$$

where $\underline{k} = \{1, 2, \dots, k\}$. A *morphism* from an embedding $f: \underline{k} \rightarrow M$ to an embedding $g: \underline{\ell} \rightarrow M$ consists of a map

$$v: \underline{k} \rightarrow \underline{\ell},$$

not necessarily injective, and a (Moore) homotopy $(\gamma_t)_{t \in [0, a]}$ from f to gv which satisfies the stickiness condition: if $\gamma_s(x) = \gamma_s(y)$ for some $s \in [0, a]$ and some $x, y \in \underline{k}$, then $\gamma_t(x) = \gamma_t(y)$ for all $t \in [s, a]$.

For homotopy theoretic purposes it is wise to replace the topological category by its topological nerve, which is a simplicial space. Therefore $\text{con}(M)$ is strictly speaking a simplicial space. Other “models” of the configuration category of M described in [3] are other simplicial spaces which are degreewise weakly equivalent to this incarnation. In all these models, $\text{con}(M)$ is a simplicial space *over* the nerve $N\text{Fin}$ of Fin , where Fin is the small category whose objects are the finite sets \underline{k} for $k \geq 0$, and all maps between these sets as morphisms. As such, $\text{con}(M)$ is a fiberwise complete Segal space over $N\text{Fin}$; this is one way to say that it has the homotopical properties that one expects from the nerve of a well-behaved topological category.

2010 *Mathematics Subject Classification*. Primary 57R19, 55P65.

This project was supported by the Humboldt foundation through a Humboldt professorship, 2012-2017.

A number of people, but especially Bill Dwyer and Ricardo Andrade, have asked whether $\mathbf{con}(M)$ as a homotopical construct is a good substitute for the topological type of M . (In the Dwyer formulation the question probably did not exactly mention $\mathbf{con}(M)$ but something closely related from the world of operads; and perhaps it was about \mathbb{R}^m rather than a general M .) More precisely, there is an inclusion map of topological (grouplike) monoids

$$\mathrm{homeo}(M) \longrightarrow \mathrm{haut}_{N\mathbf{Fin}}(\mathbf{con}(M))$$

where $\mathrm{haut}_{N\mathbf{Fin}}(\mathbf{con}(M))$ denotes the grouplike topological monoid of right derived homotopy automorphisms of $\mathbf{con}(M)$, as a simplicial space over $N\mathbf{Fin}$. One may wonder whether this map is a homotopy equivalence, or a good approximation in a weaker sense.

For example, [3, §8] implies a positive answer in a special case of an analogous question for manifolds with boundary. Namely, the space of homeomorphisms of a disk D^m relative to the boundary is contractible by the Alexander trick. The space of homotopy automorphisms of the corresponding configuration category (relative to an appropriate boundary configuration category and over the nerve of \mathbf{Fin}_* , the appropriate enlargement of \mathbf{Fin}) is also contractible.

This paper is a continuation of [10]. The main point is a translation of some of the results in [10] into a more homotopical language, specifically, the language of configuration categories. The original formulation in [10] uses instead topological posets $\mathcal{P}(L)$, $\mathcal{P}(M \setminus \partial M)$ etc., where M and L are smooth manifolds. These posets depend on a choice of Riemannian metric on M or L .

After the translation, we have a positive answer to a weak variant of the Dwyer-Andrade question. Here is a description of that answer (and the question) in a simple case. Let M be a compact smooth manifold with boundary. Let $M_- = M \setminus \partial M$. The boundary ∂M can be recovered from M_- in a homotopical sense as the homotopy link of the base point in $M/\partial M \cong M_- \cup \infty$. Therefore it is allowed to say that the homeomorphism group $\mathrm{homeo}(M_-)$ acts on the pair $(M, \partial M)$ by homotopy automorphisms. This is well known. Now we ask whether this action extends to an action of $\mathrm{haut}_{N\mathbf{Fin}}(\mathbf{con}(M_-))$ on the pair $(M, \partial M)$ by homotopy automorphisms. We obtain a positive answer, theorem 5.2.1, under some fairly severe conditions on M . Results of this type are used in [11]. They are likely to be generally useful in manifold calculus applied to spaces of smooth embeddings. They come with estimates saying that if we are happy to replace $(M, \partial M)$ by its k -type (or to kill at least some homotopy groups above level k), then we can restrict attention to configurations in M of cardinality $\leq f(k)$, where f is a fairly uncomplicated function of the variable k . I hope that these estimates are a consolation for any feelings of loss or betrayal caused by the passage from the hard Dwyer-Andrade question to an easier and weaker variant.

These estimates become trivial when $\partial M = \emptyset$. In that case it is clear that a homotopy automorphism of the category of configurations of cardinality ≤ 1 in M determines a homotopy automorphism of M . (So we can take $f(k) = 1$ for all k , in the notation above.) It is a little surprising that the case where $\partial M \neq \emptyset$ should be so much more difficult. Perhaps it *is* not much more difficult, but I have just failed to see the decisive shortcut.

Here is a short review of the notation used and the type of results proved in [10], covering the simplest cases. Suppose first that L is a smooth compact submanifold

of a smooth manifold M . Both M and L are without boundary and L is equipped with a Riemannian metric. The elements of $\mathcal{P}(L)$ are pairs (S, ρ) where S is a finite subset of L and $\rho: S \rightarrow \mathbb{R}$ is a function with positive values. There is a condition: for each $s \in S$, the exponential map is defined and regular on the closed disk of radius $\rho(s)$ in $T_s L$, and the images of these disks under the exponential maps (for each $s \in S \subset L$) are pairwise disjoint in L . Let $V_L(S, \rho) \subset L \subset M$ be the union of the images in L of the corresponding *open* balls of radius $\rho(s)$ in $T_s L$ under the exponential maps. There are results of the following form: the map

$$M \setminus L \longrightarrow \operatorname{holim}_{(S, \rho) \in \mathcal{P}(L)} M \setminus V_L(S, \rho)$$

induced by the inclusions $M \setminus L \rightarrow M \setminus V_L(S, \rho)$ is a weak equivalence, under some conditions. That map can also be written in the form

$$M \setminus L \longrightarrow \operatorname{holim} \Phi$$

where Φ is the contravariant functor $(S, \rho) \mapsto M \setminus V_L(S, \rho)$ from $\mathcal{P}(L)$ to spaces. The homotopy limit is an enriched variant.

Suppose next that M is smooth, compact, *with* boundary and equipped with a Riemannian metric; no submanifold L is specified. Define $\mathcal{P}(M \setminus \partial M)$ roughly as above in the case of L , so that elements of $\mathcal{P}(M \setminus \partial M)$ are pairs (S, ρ) where S is a finite subset of $M \setminus \partial M$ and ρ is a function with positive values on S . Again there is a condition: for each $s \in S$, the exponential map is defined and regular on the closed disk of radius $\rho(s)$ in $T_s(M \setminus \partial M)$, and the images of these disks under the exponential maps (for each $s \in S$) are pairwise disjoint in $M \setminus \partial M$. For $(S, \rho) \in \mathcal{P}(M \setminus \partial M)$ let $V(S, \rho) \subset M \setminus \partial M$ be the union of the images of the corresponding *open* balls of radius $\rho(s)$ in $T_x(M \setminus \partial M)$ under the exponential maps. There are results of the following form: the map

$$\partial M \longrightarrow \operatorname{holim}_{(S, \rho) \in \mathcal{P}(M)} M \setminus V(S, \rho)$$

induced by the inclusions $\partial M \rightarrow M \setminus V(S, \rho)$ is a weak equivalence, under some conditions. That map can also be written in the form

$$\partial M \longrightarrow \operatorname{holim} \Psi$$

where $\Psi(S, \rho) = M \setminus V(S, \rho)$ for $(S, \rho) \in \mathcal{P}(M \setminus \partial M)$. The homotopy limit is an enriched variant.

2. The language of configuration categories

The translation promised in the introduction uses Rezk's concept of a complete Segal space and the associated framework [7] in which topological categories can be viewed as objects of a model category. It is not a great challenge to recast the topological posets $\mathcal{P}(L)$, $\mathcal{P}(M \setminus \partial M)$ etc. as complete Segal spaces. Indeed their topological nerves are already complete Segal spaces; but we are going to tinker with the definitions in order to make a better connection with [3]. The more important and more challenging task for us is to make sense of the continuous functors Φ , Ψ etc. and their homotopy limits in the setting of complete Segal spaces.

2.1. Simplicial spaces, Segal spaces and complete Segal spaces. In the first few sections of [3] there are definitions of *Segal space* and *complete Segal space* with some examples and failing candidates. The central example there is the configuration category of a manifold M . This comes in many guises, well-known and less well-known. There is a variant where M has empty boundary, and a more complicated variant in the case where M has nonempty boundary. Here we only give a brief review of the definitions and examples.

The nerve construction turns small categories into simplicial sets and small topological categories into simplicial spaces. It was Graeme Segal [8], [2] who promoted the idea that the nerves and their homotopical properties are more important than the categories themselves. In that spirit Rezk [7] gave the following definition. For $r \geq 0$ and $i = 1, 2, \dots, r$ let $u_i: [1] \rightarrow [r]$ be the monotone injection defined by $u_i(0) = i - 1$ and $u_i(1) = i$. A *Segal space* is a simplicial space X such that for each $r \geq 2$ the map $(u_1^*, u_2^*, \dots, u_r^*)$ from X_r to the homotopy inverse limit of the diagram

$$X_1 \xrightarrow{d_0} X_0 \xleftarrow{d_1} X_1 \xrightarrow{d_0} \dots \quad \dots \xrightarrow{d_0} X_0 \xleftarrow{d_1} X_1,$$

with r copies of X_1 , is a weak homotopy equivalence. (In the case where X_0 is weakly contractible this simplifies to the condition that $(u_1^*, u_2^*, \dots, u_r^*)$ as a map from X_r to $(X_1)^r$ be a weak homotopy equivalence for all $r \geq 2$. This constitutes Segal's definition or interpretation of what it means for the space X_1 to have the structure of an A_∞ topological monoid with unit.) In particular the nerve of any small category is a Segal space X which also happens to be a simplicial *set*. Another important type of example: if \mathcal{C} is a topological category (category object in the category of topological spaces), and if one of the maps *source*, *target* from the space $\text{mor}(\mathcal{C})$ to the space $\text{ob}(\mathcal{C})$ is a Serre fibration, then the nerve $N\mathcal{C}$ is also a Segal space.

Equivalences between small categories are not always reflected in degreewise weak equivalences between their nerves. Indeed if \mathcal{C} is equivalent to \mathcal{D} , then there is no strong reason why $N_0\mathcal{C} = \text{ob}(\mathcal{C})$ should be weakly equivalent to $N_0\mathcal{D} = \text{ob}(\mathcal{D})$. To deal with this, Rezk introduced the concept of *Dwyer-Kan* equivalence between Segal spaces as an analogue of the classical concept of equivalence between categories, and a related concept of completeness. A Segal space Y is *complete* if the map $d_0: Y_1 \rightarrow Y_0$ alias *source* restricts to a weak equivalence from Y_1^{he} to Y_0 , where Y_1^{he} is the union of the homotopy invertible path components of Y_1 . He showed that for every Segal space X there exists a Segal space Y and a simplicial map $X \rightarrow Y$ which is a Dwyer-Kan equivalence, and where Y is complete. Moreover a Dwyer-Kan equivalence between complete Segal spaces is a degreewise weak equivalence (between simplicial spaces). *Example:* a discrete group G determines a category with one object whose endomorphism monoid is the group G . The nerve of that category is a Segal space X , but it is not complete unless G is the trivial group. The Rezk completion Y of X has the form of a constant simplicial space, $Y_r = BG$ for all $r \geq 0$.

The nerves of the topological posets $\mathcal{P}(L)$ and $\mathcal{P}(M \setminus \partial M)$ defined in section 1 are examples of complete Segal spaces. In [3] we use slightly different editions denoted $\text{con}(L)$, $\text{con}(M)$, $\text{con}(M \setminus \partial M)$ etc., for mostly bureaucratic reasons. One definition of the simplicial space $X = \text{con}(L)$ for a smooth Riemannian manifold L

with *empty* boundary is as follows. An object, alias element of X_0 , is an element (S, ρ) of $\mathcal{P}(L)$ together with a total ordering of S . There is at most one morphism between any two objects, and this happens if and only if $(S, \rho) \leq (T, \sigma)$ holds for the underlying elements (S, ρ) and (T, σ) of $\mathcal{P}(L)$. Thus X_0 is a covering space of the space of objects of $\mathcal{P}(L)$, so that the fiber over an element (S, ρ) in $\mathcal{P}(L)$ is the set of total orderings of S ; and X_1 is a covering space of the space of morphisms in $\mathcal{P}(L)$, so that the fiber over $(S, \rho) \leq (T, \sigma)$ is the product of the set of total orderings of S with the set of total orderings of T . In this way, X_0 and X_1 form object space and morphism space of a topological category (category object in the category of topological spaces). The nerve of that is $\mathbf{con}(L)$, a simplicial space; more precisely it is called the *Riemannian model* of $\mathbf{con}(L)$ in [3]. It is a Segal space (e.g. because $d_1 = \text{target}$ from X_1 to X_0 is a Serre fibration), but not a complete Segal space except in a few cases of little interest. We recover the loss by making two observations.

- The forgetful functor $\mathbf{con}(L) \rightarrow \mathcal{P}(L)$ is a Dwyer-Kan equivalence.
- $\mathbf{con}(L)$ is a *fiberwise complete Segal space* over the nerve of \mathbf{Fin} (explanation follows).

Here \mathbf{Fin} is the small category whose objects are the finite sets $\underline{n} = \{1, 2, \dots, n\}$ for $n \geq 0$ with all maps between these sets as morphisms. There is an obvious forgetful functor from $\mathbf{con}(M)$ in the above definition to $N\mathbf{Fin}$, the nerve of \mathbf{Fin} . By saying that $\mathbf{con}(L) \rightarrow N\mathbf{Fin}$ is a fiberwise complete Segal space we mean that the resulting commutative square

$$\begin{array}{ccc} X_1^{\text{he}} & \xrightarrow{d_0} & X_0 \\ \downarrow & & \downarrow \\ Y_1^{\text{he}} & \xrightarrow{d_0} & Y_0 \end{array}$$

(where $X = \mathbf{con}(L)$ and $Y = N\mathbf{Fin}$) is a weak homotopy pullback square.

In the case where L has nonempty boundary, there is a more complicated definition of $\mathcal{P}(L)$ and a related definition of $\mathbf{con}(L)$. The elements of $\mathcal{P}(L)$ are pairs (S, ρ) where S is a finite subset of $L \setminus \partial L$ and ρ is a function from $S \sqcup \partial L$ to the positive reals, locally constant on ∂L and subject to a few more conditions.

- For each $s \in S$, the exponential map \exp_s at s is defined and regular on the disk of radius $\rho(s)$ about the origin in $T_s L$.
- The (boundary-normal) exponential map is defined and regular on the set of all tangent vectors $v \in T_z L$ where $z \in \partial L$, where the vector v is inward perpendicular to $T_z \partial L$ and $|v| \leq \rho(z)$.
- The images in L of these disks and the image of this band under the exponential map(s) are pairwise disjoint.

For a pair (S, ρ) satisfying these conditions, let $V(S, \rho) \subset L$ be the union of the open balls of radius $\rho(s)$ about elements $s \in S$ and the open collar on ∂L determined the normal distance function $\rho|_{\partial L}$. (Sometimes we write $V_L(S, \rho)$ instead of $V(S, \rho)$, for example if L comes as a smooth submanifold of another smooth manifold M .) In this case there is a reference map $\mathbf{con}(L) \rightarrow N\mathbf{Fin}_*$, where \mathbf{Fin}_* is the following category. Objects are the finite sets $[n] = \{0, 1, \dots, n\}$ for $n \geq 0$ which we view as sets with a base point 0, and the morphisms are all base-point preserving maps between these. Both $\mathbf{con}(L)$ and $N\mathbf{Fin}_*$ are Segal spaces and the reference map $\mathbf{con}(L) \rightarrow N\mathbf{Fin}_*$ makes $\mathbf{con}(L)$ into a fiberwise complete Segal space over $N\mathbf{Fin}_*$.

There are many alternative descriptions of $\text{con}(L)$ in [3]; each of these can be related to the above definition by a chain of degree-wise equivalences over $N\text{Fin}$ or over $N\text{Fin}_*$, as appropriate. (Actually L is typically called M in [3] and now we shall adopt that habit.) One of them, the *particle model*, deserves to be mentioned here because it does not require a Riemannian metric, or even a smooth structure, and has very good naturality properties. It is probably due to [1]. Suppose to begin with that M is a topological manifold with empty boundary. Let $k \in \mathbb{N}$. The space of maps from \underline{k} to M comes with an obvious stratification. There is one stratum for each equivalence relation η on \underline{k} . The points of that stratum are precisely the maps $\underline{k} \rightarrow M$ which can be factorized as projection from \underline{k} to \underline{k}/η followed by an injection of \underline{k}/η into M .

Now we construct a topological category whose object space is

$$(2.1.1) \quad \coprod_{k \geq 0} \text{emb}(\underline{k}, M).$$

By a morphism from $f \in \text{emb}(\underline{k}, M)$ to $g \in \text{emb}(\underline{\ell}, M)$ we mean a pair consisting of a map $v: \underline{k} \rightarrow \underline{\ell}$ and a *reverse exit path* γ from f to gv in the stratified space of all maps from \underline{k} to M . (In more detail: γ is a path $[0, a] \rightarrow \text{map}(\underline{k}, M)$ for some $a \geq 0$, and the reverse exit path property means that if $\gamma_t(x) = \gamma_t(y)$ for some $t \in [0, a]$ and $x, y \in S$, then $\gamma_s(x) = \gamma_s(y)$ for all $s \in [t, a]$. Note that f is injective but gv need not be injective since v is not required to be injective.) The space of all morphisms is therefore a coproduct with one summand for each morphism $v: \underline{k} \rightarrow \underline{\ell}$ in Fin , where that summand consists of triples (f, g, γ) as above: $f \in \text{emb}(\underline{k}, M)$, $g \in \text{emb}(\underline{\ell}, M)$ and γ a reverse exit path from gv to f . Composition of morphisms is obvious. The nerve of this category is a fiberwise complete Segal space over $N\text{Fin}$.

In the case of a topological manifold M with nonempty boundary, the definition of $\text{con}(M)$ along similar lines is slightly more complicated, but we need it. The space of maps from \underline{k} to M comes with a stratification. There is one stratum for each pair (S, η) where $S \subset \underline{k}$ and η is an equivalence relation on \underline{k} such that S is a union of equivalence classes. The points of that stratum are the maps $\underline{k} \rightarrow M$ taking S to ∂M and the complement of S to $M \setminus \partial M$, and which can be factored as projection from \underline{k} to \underline{k}/η followed by an injection of \underline{k}/η into M . Now we construct a topological category whose object space is

$$(2.1.2) \quad \coprod_{k \geq 0} \text{emb}(\underline{k}, M \setminus \partial M).$$

A morphism from $f \in \text{emb}(\underline{k}, M \setminus \partial M)$ to $g \in \text{emb}(\underline{\ell}, M \setminus \partial M)$ is a pair consisting of a morphism $v: [k] \rightarrow [\ell]$ in Fin_* and a Moore path $\gamma = (\gamma_t)_{t \in [0, a]}$ in $\text{map}(\underline{k}, M)$ which is a reverse exit path. It is required to satisfy $\gamma_0 = f$ and $\gamma_a(x) = g(v(x))$ if $v(x) \in \underline{\ell}$, but $\gamma_a(x) \in \partial M$ if $v(x) = 0$. Composition of morphisms is almost obvious. The nerve of this topological category is a fiberwise complete Segal space over $N\text{Fin}_*$ which we can regard as an alternative definition or description of $\text{con}(M)$.

2.2. Functors as maps between simplicial spaces. A *map* between complete Segal spaces X and Y is a simplicial map $f: X \rightarrow Y$. Such an f can also be regarded as a functor from X to Y . The map f is considered to be a *weak equivalence* if each $f_r: X_r \rightarrow Y_r$ is a weak equivalence of spaces. For many purposes it is useful to have a notion of *space* of maps from X to Y which is functorial in the two variables X and Y and takes weak equivalences to weak equivalences. Such a

concept exists and is called the *derived* space of maps from X to Y , and denoted

$$\mathbb{R}\mathrm{map}(X, Y) .$$

To define this we do not need to know or assume that X and Y are complete Segal spaces. It suffices to know that they are simplicial spaces. It suffices to have a decision as to which simplicial maps between simplicial spaces are to be called *weak equivalences* (namely, those which are degreewise weak equivalences of spaces).

Therefore we switch briefly to the general setting where X and Y are contravariant functors from a (small, discrete) category \mathcal{C} to the category of spaces. (The example to keep in mind is $\mathcal{C} = \Delta$.) Let \mathcal{D} be the category of such functors from \mathcal{C} to spaces, where morphisms alias *maps* are natural transformations. A map $f: X \rightarrow Y$ in \mathcal{D} is a weak equivalence if $f_c: X(c) \rightarrow Y(c)$ is a weak equivalence of spaces for each object c in \mathcal{C} . It is straightforward to define a space $\mathrm{map}(X, Y)$, for example as the geometric realization of the simplicial set where a k -simplex is a map from $X \times \Delta^k$ to Y , where Δ^k is the geometric k -simplex (a space). We look for a definition of $\mathbb{R}\mathrm{map}(X, Y)$, the derived mapping space. There are two well-known options.

- Dwyer and Kan [4] have a definition of $\mathbb{R}\mathrm{map}(X, Y)$ in an extremely general setting where X and Y are objects in a category \mathcal{D} with a subcategory of so-called weak equivalences, subject to some mild conditions. Their definition of $\mathbb{R}\mathrm{map}(X, Y)$ is big in the sense that it can be a simplicial class rather than a simplicial set if \mathcal{D} is not small.
- For the category \mathcal{D} (as defined above, category of contravariant functors from \mathcal{C} to spaces) we have a subcategory of *weak equivalences* and a preferred action of the category of simplicial sets on \mathcal{D} , given by the ordinary degreewise product of simplicial sets with simplicial spaces. There are a few standard ways to enhance these data to the structure of a Quillen simplicial model category [6], [5]. (For us the preferred choice is the one where a map between simplicial spaces is considered to be a fibration if it is a degreewise Serre fibration.) Then $\mathbb{R}\mathrm{map}(X, Y)$ can be defined as $\mathrm{map}(X^\flat, Y^\sharp)$ where X^\flat is a cofibrant replacement of X and Y^\sharp is a fibrant replacement of Y . To achieve strict functoriality one should use functorial replacements, so that $X \mapsto X^\flat$ is a functor with a natural transformation to the identity by weak equivalences, and $Y \mapsto Y^\sharp$ is a functor with a natural transformation from the identity by weak equivalences.

It is a special case of a result in [4] that these two definitions of $\mathbb{R}\mathrm{map}(X, Y)$ agree up to a chain of weak equivalences. (This has the consequence that $\mathbb{R}\mathrm{map}(X, Y)$ according to the second definition is largely independent of the choices required there.)

Returning to simplicial spaces X and Y , we conclude that we have a few good definitions of a derived mapping space $\mathbb{R}\mathrm{map}(X, Y)$, since a simplicial space is a contravariant functor from Δ to spaces. More generally, suppose that Z is a fixed simplicial set, and let X, Y be simplicial spaces over Z , that is to say, simplicial spaces equipped with reference maps p_X and p_Y to Z , respectively. By $\mathbb{R}\mathrm{map}_Z(X, Y)$ we mean the fiber of the map $\mathbb{R}\mathrm{map}(X, Y) \rightarrow \mathrm{map}(X, Z)$ given by composition with p_Y over the point determined by p_X . Perhaps it is worth pointing out that $\mathrm{map}(X, Z)$ is a set, alias discrete space. We are also using the fact that $p_Y: Y \rightarrow Z$ extends uniquely to a map $Y^\sharp \rightarrow Z$.

3. Derived section spaces and the shift construction

What we are after in this section is a description of functors such as the functor Ψ in section 1, and their homotopy inverse limits, in terms of not much more than the source category. In the setting of section 1 the source category would be $\mathcal{P}(M \setminus \partial M)$, but it is better for us to use the variant $\text{con}(M \setminus \partial M)$ with the reference functor from there to Fin .

3.1. Derived section spaces. Let $p: E \rightarrow X$ be a map between simplicial spaces. Choose a factorization

$$E \longrightarrow E^\sharp \xrightarrow{p^\sharp} X$$

of p where the first arrow is a weak equivalence and p^\sharp is a fibration. Choose a weak equivalence $X^b \rightarrow X$ where X^b is cofibrant.

DEFINITION 3.1.1. The *derived section space* of p , denoted $\mathbb{R}\Gamma(p)$, is the space of lifts as in the diagram

$$\begin{array}{ccc} & E^\sharp & \\ & \downarrow p^\sharp & \\ X^b & \xrightarrow{\quad} & X \end{array}$$

Using functorial replacements p^\sharp and X^b of p and X is a good idea.

3.2. The shift construction. Let X be a Segal space and let A be any simplicial space. Rezk has a definition of an *internal* mapping object X^A which is as follows (in a possibly simplified form which I hope is good enough here). Put

$$(X^A)_n := \mathbb{R}\text{map}(\Delta[n] \times A, X)$$

where $\Delta[n]$ is the simplicial set freely generated by one element in degree n , so that the geometric realization $|\Delta[n]|$ is the standard geometric n -simplex Δ^n .

PROPOSITION 3.2.1. (Rezk.) *If X is a Segal space, then X^A is a Segal space; if X is a complete Segal space, then X^A is a complete Segal space.* \square

EXAMPLE 3.2.2. If X and A are both complete Segal spaces, then X^A should be viewed as the *category* of functors from A to X . In particular $(X^A)_0 \cong \mathbb{R}\text{map}(A, X)$ should be viewed as the (derived) space of functors from A to X and $(X^A)_n = \mathbb{R}\text{map}(\Delta[n] \times A, X)$ should be viewed as the (derived) space of constellations

$$G_0 \leftarrow G_1 \leftarrow \cdots \leftarrow G_n$$

where G_0, G_1, \dots, G_n are functors from A to X and the arrows connecting them are natural transformations. (In particular, if A is the simplicial space which has a single point in each degree, then we recover the idea that $(X^A)_n \simeq X_n$ is the derived space of functors from $[n]^{\text{op}}$ to X .)

DEFINITION 3.2.3. There is a functor $\sigma: \text{Fin} \rightarrow \text{Fin}$ given by disjoint sum with a singleton. In more detail, σ is given by $\{1, 2, \dots, k\} \mapsto \{1, 2, \dots, k, k+1\}$ on objects, and for a morphism $f: \underline{k} \rightarrow \underline{\ell}$ the morphism $\sigma(f)$ is given by $\sigma(f)(x) = f(x)$ for $x \leq k$ and $\sigma(f)(k+1) = \ell+1$. The standard inclusions of $\{1, 2, \dots, k\}$ in $\{1, 2, \dots, k, k+1\}$ define a natural transformation $u: \text{id} \rightarrow \sigma$ between endofunctors

of \mathbf{Fin} . Together, σ and u determine a map from $\Delta[1] \times N\mathbf{Fin}$ to $N\mathbf{Fin}$, or in adjoint form, a map of Segal spaces

$$N\mathbf{Fin} \rightarrow N\mathbf{Fin}^{\Delta[1]}.$$

Since $N\mathbf{Fin}^{\Delta[1]}$ is a simplicial set and at the same time a Segal space, it is (isomorphic) to the nerve of a small category, and this is also easy to see directly.

Let X be a simplicial space over $N\mathbf{Fin}$. Let $E^\sigma(X)$ be the simplicial space defined by the pullback square of simplicial spaces and simplicial sets

$$\begin{array}{ccc} E^\sigma(X) & \longrightarrow & X^{\Delta[1]} \\ \downarrow & & \downarrow \\ N\mathbf{Fin} & \xrightarrow{(\sigma, u)} & N\mathbf{Fin}^{\Delta[1]} \end{array}$$

There is a map $\psi_X : E^\sigma(X) \rightarrow X$ over $N\mathbf{Fin}$ given by composing $E^\sigma(X) \rightarrow X^{\Delta[1]}$ from the defining pullback square with the map

$$X^{\Delta[1]} \rightarrow X^{\Delta[0]} \cong X$$

determined by the map $\Delta[0] \rightarrow \Delta[1]$ which takes the preferred generator in degree 0 to d_0 of the preferred generator in degree 1.

Let us see how the derived section space $\mathbb{R}\Gamma(\psi_X)$ depends on X . (Reason for being interested in $\mathbb{R}\Gamma(\psi_X)$: it is a good homotopical substitute for ∂M when X is $\text{con}(M \setminus \partial M)$ for a compact smooth manifold M .) We would like to say that homotopy automorphisms of X as a simplicial space over $N\mathbf{Fin}$ induce homotopy automorphisms of $\mathbb{R}\Gamma(\psi_X)$. The following is a slightly pedestrian justification. Abbreviate $W(X) := \mathbb{R}\Gamma(\psi_X)$. Let $v : X \rightarrow Y$ be a weak equivalence between simplicial spaces. This determines a commutative square of simplicial spaces

$$\begin{array}{ccc} E^\sigma(X) & \xrightarrow{E^\sigma(v)} & E^\sigma(Y) \\ \downarrow \psi_X & & \downarrow \psi_Y \\ X & \xrightarrow{v} & Y \end{array}$$

Define $W(v)$ as the simplicial set of triples (s, t, h) where s and t are derived sections of ψ_X and ψ_Y respectively, and h is a homotopy connecting $E^\sigma(v)^\sharp \circ s$ to $t \circ v^\flat$. (More precisely a k -simplex of $W(v)$ is a family of such triples (s, t, h) parameterized by the geometric k -simplex Δ^k .) There are forgetful weak equivalences

$$W(X) \longleftarrow W(v) \longrightarrow W(Y)$$

which are also Kan fibrations. This is already enough to establish a great deal of naturality for the construction $X \mapsto W(X)$. Namely, for a fixed X , simplicial space over $N\mathbf{Fin}$, let \mathcal{C}_X be a small subcategory of the category of simplicial spaces over $N\mathbf{Fin}$ with the following properties.

- X is an object of \mathcal{C}_X .
- Every object of \mathcal{C}_X is weakly equivalent to X (in the category of simplicial spaces over $N\mathbf{Fin}$).
- If Y belongs to \mathcal{C}_X , then $Y \times \Delta^k$ also belongs to \mathcal{C}_X for every $k \geq 0$.

- The morphisms in \mathcal{C}_X between any two objects of \mathcal{C}_X are precisely the weak equivalences between these two in the category of simplicial spaces over $N\text{Fin}$.
- \mathcal{C}_X is closed under (some) functorial cofibrant replacement in the category of simplicial spaces. (At this point, some decisions must be made on the meaning of *space* and on preferred model category structures on the category of spaces and on the category of simplicial spaces. Let us say, for example, that *space* just means topological space and that we use the standard model category structure on the category of spaces where *fibration* means *Serre fibration* and the weak equivalences are the maps which are classically called weak equivalences. There is then a preferred model category structure on the category of simplicial spaces where the weak equivalences are the simplicial maps which are degreewise weak equivalences in the category of spaces, and the fibrations are degreewise fibrations in the category of spaces. This is a good choice for our purposes.)

It is well known that the classifying space $B\mathcal{C}_X$ is then a correct model for the spaces $B\text{haut}_{N\text{Fin}}(X)$, where $\text{haut}_{N\text{Fin}}(X)$ is the union of the homotopy invertible components of $\mathbb{R}\text{map}_{N\text{Fin}}(X, X)$. A diagram of the shape

$$X(0) \xleftarrow{v(1)} X(1) \xleftarrow{v(2)} \cdots \xleftarrow{v(k)} X(k)$$

in \mathcal{C}_X determines a space $W(v(1), \dots, v(k))$, the inverse limit of the diagram of simplicial sets

$$W(v(1)) \rightarrow W(X(1)) \leftarrow W(v(2)) \rightarrow W(X(2)) \leftarrow \cdots \rightarrow W(X(k-1)) \leftarrow W(v(k)).$$

This comes with forgetful projections to the $W(X(i))$ which are weak equivalences and Kan fibrations. Now we have the following projection map.

$$(3.2.1) \quad \text{hocolim}_{(v(1), \dots, v(k))} W(v(1), \dots, v(k)) \longrightarrow \text{hocolim}_{(v(1), \dots, v(k))} \star$$

These homotopy colimits are taken over the category where an object is a contravariant functor from the ordered set $[k] = \{0, 1, \dots, k\}$ to \mathcal{C}_X , for some k , and a morphism from $v: [k]^{\text{op}} \rightarrow \mathcal{C}_X$ to $v': [\ell]^{\text{op}} \rightarrow \mathcal{C}_X$ is a monotone injective map $u: [k] \rightarrow [\ell]$ such that $v'u = v$. The target of the map (3.2.1) is still an incarnation of $B\text{haut}_{N\text{Fin}}(X)$ and the map itself is a quasi-fibration with fibers weakly equivalent to $W(X)$. Therefore we can say that the map (3.2.1) determines a classifying map

$$(3.2.2) \quad B\text{haut}_{N\text{Fin}}(X) \longrightarrow B\text{haut}(W(X)) = B\text{haut}(\mathbb{R}\Gamma(\psi_X))$$

where $\text{haut}(W(X))$ is the union of the homotopy invertible path components of $\mathbb{R}\text{map}(W(X), W(X))$.

DEFINITION 3.2.4. Definition 3.2.3 has a variant in which Fin_* takes the place of Fin . This is fairly straightforward. It begins with a functor $\sigma: \text{Fin}_* \rightarrow \text{Fin}_*$ given by disjoint sum with a singleton. In more detail, σ is given by $[k] \mapsto [k+1]$ on objects, and for a morphism $f: [k] \rightarrow [\ell]$ (based map) the morphism $\sigma(f)$ is given by $\sigma(f)(x) = f(x)$ for $x \leq k$ and $\sigma(f)(k+1) = \ell+1$. The standard inclusions of $[k]$ in $[k+1]$ define a natural transformation $u: \text{id} \rightarrow \sigma$ between endofunctors of Fin_* . For a simplicial space X over $N\text{Fin}_*$ we define $\psi_X: E^\sigma(X) \rightarrow X$ and $\mathbb{R}\Gamma(\psi_X)$ as in definition 3.2.3, *mutatis mutandis*.

3.3. Homotopy inverse limits as derived section spaces. The homotopy limit of Ψ in [10, §2.1] was defined using the Bousfield-Kan formula, i.e., as Tot of a certain cosimplicial space $[r] \mapsto \Gamma_r(\Psi)$. Here $\Gamma_r(\Psi)$ is the section space of a fiber bundle $E_r^!(\Psi) \rightarrow N_r\mathcal{P}(M \setminus \partial M)$ such that the fiber over

$$((S_0, \rho_0) \geq (S_1, \rho_1) \geq \cdots \geq (S_r, \rho_r))$$

is $M \setminus V(S_r, \rho_r)$.

From a model category point of view we should have proceeded differently. The first step should have been to introduce $E_r(\Psi)$, total space of a fiber bundle on $N_r\mathcal{P}(M \setminus \partial M)$ such that the fiber over

$$((S_0, \rho_0) \geq (S_1, \rho_1) \geq \cdots \geq (S_r, \rho_r))$$

is $M \setminus V(S_0, \rho_0)$. Note the difference between $E_r^!(\Psi)$ and $E_r(\Psi)$. Now $[r] \mapsto E_r(\Psi)$ is a simplicial space and the projections $E_r(\Psi) \rightarrow N_r\mathcal{P}(M \setminus \partial M)$ make up a simplicial map $p: E(\Psi) \rightarrow N\mathcal{P}(M \setminus \partial M)$. It is easy to see that $E(\Psi)$ is a complete Segal space like $N\mathcal{P}(M \setminus \partial M)$, although this will not be used explicitly in the following. Think of $E(\Psi)$ as the Grothendieck construction (also known as transport category) of the contravariant functor Ψ . This suggests the definition

$$(3.3.1) \quad \text{holim } \Psi := \mathbb{R}\Gamma(p: E(\Psi) \rightarrow N\mathcal{P}(M \setminus \partial M)).$$

Now we need to show that this is in agreement with the definition of $\text{holim } \Psi$ used in [10, §2.1]. In this section we have favored the model structure on the category of simplicial spaces where a morphism is a fibration if it is degreewise a Serre fibration (and a weak equivalence if it is degreewise a weak equivalence). But the definition of $\text{holim } \Psi$ given in [10, §2.1] is more easily understood in terms of the Reedy model structure on the category of simplicial spaces. It was already pointed out in [10, 1.1.3] that the simplicial space $N\mathcal{P}(M \setminus \partial M)$ is Reedy cofibrant. The map $p: E(\Psi) \rightarrow N\mathcal{P}(M \setminus \partial M)$ is not (claimed to be) a Reedy fibration, but the definition of $\text{holim } \Psi$ in [10, §2.1] contains a well-concealed suggestion for a replacement by a Reedy fibration. Let

$$x = ((S_0, \rho_0) \geq (S_1, \rho_1) \geq \cdots \geq (S_r, \rho_r))$$

be a point in $N_r\mathcal{P}(M \setminus \partial M)$. Let $F_{r,x}$ be the space of maps from Δ^r to M satisfying the condition that, for every monotone injective $u: [t] \rightarrow [r]$, the corresponding face of Δ^r is taken to $M \setminus V(S_{u(t)}, \rho_{u(t)})$. Let

$$E_r^\sharp(\Psi) \rightarrow N_r(\mathcal{P}(M \setminus \partial M))$$

be the fibration such that the fiber over x is $F_{r,x}$. There is an inclusion

$$E_r(\Psi) \rightarrow E_r^\sharp(\Psi) ;$$

indeed $F_{r,x} \cap E_r(\Psi)$ is precisely the subspace of the constant elements in $F_{r,x}$. Moreover it is easy to see that $E_r^\sharp(\Psi)$ is a simplicial space again. In the factorization

$$E(\Psi) \hookrightarrow E^\sharp(\Psi) \longrightarrow N\mathcal{P}(M \setminus \partial M)$$

of p , the first map is a weak equivalence and the second is a Reedy fibration. Therefore it is allowed to define $\text{holim } \Psi$ as the space of sections of the map of simplicial spaces

$$(3.3.2) \quad E^\sharp(\Psi) \longrightarrow N\mathcal{P}(M \setminus \partial M)$$

and this is exactly the definition of $\text{holim } \Psi$ given in [10, §2.1].

Now it is also clear how we can relate the older definition of $\text{holim } \Psi$ to the alternative definition (3.3.1). Namely, we pass from the honest section space of (3.3.2) to the derived section space of (3.3.2) in a possibly different model category structure (with the same weak equivalences), and compare that to the derived section space (3.3.1).

EXAMPLE 3.3.1. One of the good things that we get from this section is an identification of $\text{holim } \Psi$ in section 1 or [10, §2.1] with $\mathbb{R}\Gamma(\psi_X)$ of definition 3.2.3, where X is $\text{con}(M \setminus \partial M)$, a fiberwise complete Segal space over $N\text{Fin}$. There are a few simple steps to this conversion.

- (i) We start with $X = N\mathcal{P}(M \setminus \partial M)$ and the map $p: E \rightarrow X$ of simplicial spaces where $E_r = E_r(\Psi)$ is the total space of a bundle on X_r such that the fiber over a point $((S_0, \rho_0) \geq \cdots \geq (S_r, \rho_r))$ is $M \setminus V(S_0, \rho_0)$. By definition and by the foregoing discussion, the homotopy limit $\text{holim } \Psi$ in section 1 is $\mathbb{R}\Gamma(p)$, which can be thought of as the space of sections of $p^\sharp: E^\sharp \rightarrow X$, where p^\sharp is a Reedy fibration replacing p .
- (ii) We modify X , E and E^\sharp in (i) by choosing total orderings for all configurations in sight. The new X is now entitled to the name $\text{con}(M \setminus \partial M)$ and it is a simplicial space over $N\text{Fin}$. We obtain a new $\mathbb{R}\Gamma(p)$, space of sections of the new $p^\sharp: E^\sharp \rightarrow X$. The space $\mathbb{R}\Gamma(p)$ in (i) maps to the new version $\mathbb{R}\Gamma(p)$ here in (ii) by a weak equivalence.
- (iii) We keep X as in (ii) but make some small changes to E and E^\sharp . The new E_r is the total space of a fiber bundle on X_r such that the fiber over a string $((S_0, \rho_0) \geq \cdots \geq (S_r, \rho_r)) \in X_r$, where the sets S_0, \dots, S_r are totally ordered, is the space of pairs (z, ε) where $z \in M \setminus V(S_0, \rho_0)$ and ε is a positive real number which is less than the distance from z to ∂M and less than the distance from z to the closure of $V(S_0, \rho_0)$. The new E is entitled to the name $E^\sigma(X)$. We obtain a new variant of $p: E \rightarrow X$ which is entitled to the name $\psi_X: E^\sigma(X) \rightarrow X$. We obtain a new variant of $\mathbb{R}\Gamma(p)$. This is entitled to the name $\mathbb{R}\Gamma(\psi_X)$ where $X = \text{con}(M \setminus \partial M)$.

Similar good things can be said about $\text{holim } \Psi$ in [10, §3.2]. It can be identified with $\mathbb{R}\Gamma(\psi_X)$ of definition 3.2.4, where X is $\text{con}(M \setminus \partial_1 M)$, a fiberwise complete Segal space over $N\text{Fin}_*$. To recall some of the details: M is a smooth compact Riemannian manifold with boundary and corners in the boundary, so that

$$\partial M = \partial_0 M \cup \partial_1 M$$

where $\partial\partial_0 M = \partial\partial_1 M = \partial_0 M \cap \partial_1 M$. The topological poset $\mathcal{P}(M \setminus \partial_1 M)$ has elements (S, ρ) where S is a finite subset of $M \setminus \partial M$ and $\rho: S \sqcup \partial_0 M \rightarrow \mathbb{R}$ is a function with positive values, locally constant on $\partial_0 M$. There are some smallness conditions on ρ as usual. In addition it is required that the Riemannian metric on M be a product metric near $\partial_1 M$ (product of a Riemannian metric on $\partial_1 M$ and the standard metric on a closed interval). For $(S, \rho) \in \mathcal{P}(M \setminus \partial M)$ the set $V(S, \rho)$ is defined as an open subset of $M \setminus \partial_1 M$, but the functor Ψ is defined by $(S, \rho) \mapsto M \setminus V(S, \rho)$. As a result there is a map

$$\partial_1 M \rightarrow \text{holim } \Psi$$

induced by the inclusion $\partial_1 M \rightarrow M \setminus V(S, \rho)$. This is a weak equivalence under some (rather severe) conditions on M .

4. Twisted arrow construction and shifting

In this section the construction in [10, §4.1] of a functor Θ from the twisted arrow construction on the topological poset $\mathcal{P}(M \setminus \partial_1 M)$ to spaces will be generalized. In the general form it is applicable to a simplicial space X over $N\text{Fin}_*$ which replaces the nerve of $\mathcal{P}(M \setminus \partial_1 M)$.

4.1. Twisted arrow construction on simplicial spaces. The twisted arrow construction on a simplicial space X is $\text{tw}(X) := X \circ \beta$, where $\beta: \Delta \rightarrow \Delta$ is the functor $[n] \mapsto [2n + 1]$. More precisely, Δ is the category of totally ordered nonempty finite sets and order-preserving maps, or the equivalent full subcategory with objects $[n]$ for $n \geq 0$, and β is the functor which takes a totally ordered nonempty finite set S to $S \sqcup S^{\text{op}}$ (with the concatenated total ordering where $a < b$ if $a \in S \subset S \sqcup S^{\text{op}}$ and $b \in S^{\text{op}} \subset S \sqcup S^{\text{op}}$).

The inclusions $S \rightarrow S \sqcup S^{\text{op}}$ define a natural transformation $e: \text{id} \rightarrow \beta$. This induces a simplicial map $\text{tw}(X) \rightarrow X$ of simplicial spaces which is entitled to the name *source*. Example: If $X = N\mathcal{C}$ for a small category \mathcal{C} , then $\text{tw}(X) = N(\text{tw}(\mathcal{C}))$ where $\text{tw}(\mathcal{C})$ is the twisted arrow category of \mathcal{C} . (An object of $\text{tw}(\mathcal{C})$ is a morphism in \mathcal{C} and a morphism in $\text{tw}(\mathcal{C})$ is a commutative diagram

$$\begin{array}{ccc} a & \longrightarrow & b \\ \downarrow & & \uparrow \\ c & \longrightarrow & d \end{array}$$

in \mathcal{C} , where the top row is the source object in $\text{tw}(\mathcal{C})$ and the bottom row is the target object.) The canonical map $\text{tw}(X) \rightarrow X$ is then the map of nerves induced by the forgetful functor which takes an object in $\text{tw}(\mathcal{C})$, alias morphism $a \rightarrow b$ in \mathcal{C} , to its source a .

4.2. More shifting. As mentioned in definition 3.2.4, the functor σ from Fin to Fin and the natural transformation $u: \text{id} \rightarrow \sigma$ of definition 3.2.3 extend in a straightforward way to a functor $\text{Fin}_* \rightarrow \text{Fin}_*$ and a natural transformation from $\text{id}: \text{Fin}_* \rightarrow \text{Fin}_*$ to σ . These are still denoted σ and u , respectively. Here we need another variant consisting of a functor $\tau: \text{tw}(\text{Fin}_*) \rightarrow \text{tw}(\text{Fin}_*)$ and a natural transformation v from the identity on $\text{tw}(\text{Fin}_*)$ to τ .

DEFINITION 4.2.1. On objects, $\tau: \text{tw}(\text{Fin}_*) \rightarrow \text{tw}(\text{Fin}_*)$ is defined by

$$\tau(f: [m] \rightarrow [n]) := (g: [m + 1] \rightarrow [n])$$

where $g(x) = f(x)$ for $x \in [m]$ and $g(m + 1) := 0$. The remaining details are settled in such a way that $F_s \tau = \sigma F_s$ and $F_t \tau = F_t$, where $F_s, F_t: \text{tw}(\text{Fin}_*) \rightarrow \text{Fin}_*$ are the functors given by source and target, respectively.

For a simplicial space Y over $\text{tw}(N\text{Fin}_*) = N(\text{tw}(\text{Fin}_*))$ let $E^\tau(Y)$ be the simplicial space defined by the pullback square of simplicial spaces and simplicial sets

$$\begin{array}{ccc} E^\tau(Y) & \longrightarrow & Y^{\Delta[1]} \\ \downarrow & & \downarrow \\ N\text{tw}(\text{Fin}_*) & \xrightarrow{(\tau, v)} & (N\text{tw}(\text{Fin}_*))^{\Delta[1]} \end{array}$$

There is a map $\theta_Y: E^\tau(Y) \rightarrow Y$ over $N\text{tw}(\text{Fin}_*)$ given by composing the map $E^\tau(Y) \rightarrow Y^{\Delta[1]}$ from the defining pullback square with the map

$$Y^{\Delta[1]} \rightarrow Y^{\Delta[0]} \cong Y$$

determined by the map $\Delta[0] \rightarrow \Delta[1]$ which takes the preferred generator in degree 0 to d_0 of the preferred generator in degree 1.

Reasoning as in section 3.2, we find that the space of homotopy automorphisms $\text{haut}_{N\text{tw}(\text{Fin}_*)}(Y) = \mathbb{R}\text{map}_{N\text{tw}(\text{Fin}_*)}^\times(Y, Y)$ acts on the derived section space $\mathbb{R}\Gamma(\theta_Y)$. More precisely this action is constructed as a map

$$B\text{haut}_{N\text{tw}(\text{Fin}_*)}(Y) \longrightarrow B\text{haut}(\mathbb{R}\Gamma(\theta_Y)).$$

More specifically, if $Y = \text{tw}(X)$ for a simplicial space X over $N\text{Fin}_*$, then we also have an obvious map $B\text{haut}_{N\text{Fin}_*}(X) \rightarrow B\text{haut}_{N\text{tw}(\text{Fin}_*)}(Y)$ which we can compose with the above to conclude that $\text{haut}_{N\text{Fin}_*}(X)$ acts on $\mathbb{R}\Gamma(\theta_Y)$.

EXAMPLE 4.2.2. Of particular interest is the case where $Y = \text{tw}(X)$ and X is the simplicial space $\text{con}(M \setminus \partial_1 M)$, for a smooth compact manifold M with corners, $\partial M = \partial_0 M \cup \partial_1 M$ etc., as in [10, §4.1]. In that case the derived section space $\mathbb{R}\Gamma(\theta_{\text{tw}(X)})$ can be identified with the space $\text{holim } \Theta$ in [10, §4.1]. The reasoning is analogous to that in example 3.3.1. The commutative diagram

$$(4.2.1) \quad \begin{array}{ccc} \partial\partial_1 M & \xrightarrow{\text{inclusion}} & \partial_1 M \\ \downarrow & & \downarrow \\ \text{holim } \Theta & \longrightarrow \text{holim } \Psi \circ F_s \longleftarrow & \text{holim } \Psi \end{array}$$

of [10, §4.1] can be recast as

$$\begin{array}{ccc} \partial\partial_1 M & \xrightarrow{\text{inclusion}} & \partial_1 M \\ \downarrow & & \downarrow \\ \mathbb{R}\Gamma(\theta_{\text{tw}(X)}) & \longrightarrow \mathbb{R}\Gamma(F_s^* \psi_X) \longleftarrow & \mathbb{R}\Gamma(\psi_X) \end{array}$$

where $F_s^* \psi_X$ is defined by a (degreewise) homotopy pullback square of simplicial spaces and simplicial maps

$$\begin{array}{ccc} F_s^* E^\sigma(X) & \longrightarrow & E^\sigma(X) \\ F_s^* \psi_X \downarrow & & \downarrow \psi_X \\ \text{tw}(X) & \xrightarrow{F_s} & X. \end{array}$$

In the remaining sections of this article it will be useful for us to know that $\text{holim } \Theta$ has a locality property. Briefly, its homotopy type depends only on an arbitrarily small neighborhood U of $\partial_0 M$ in M . This is established in section A by reduction to a weaker statement of that type proved in [10, §4.2]. Doubts about the stronger form raised in [10, §4.2] have turned out to be unjustified.

5. Configuration categories and homotopy automorphisms

It is easy to reformulate [10, Thm. 2.1.1] and [10, Thm. 3.2.1] in the language of configuration categories. In itself that may not be worthwhile, but we can make something new out of that by combining with the observation that the functor Ψ in these theorems, translated to the configuration category setting, admits a description in terms of the source category. That was already mentioned in sections 3.2 and 3.3.

5.1. Boundary of a compact manifold as a homotopy link. For a smooth compact M , let $\text{holink}(M/\partial M, \star)$ be the space of paths $w: [0, 1] \rightarrow M/\partial M$ which satisfy $w^{-1}(\star) = \{0\}$, with the compact-open topology. Evaluation at $1 \in [0, 1]$ gives a map

$$(5.1.1) \quad q_M: \text{holink}(M/\partial M, \star) \longrightarrow M \setminus \partial M.$$

It is well known, and it will be made precise in a moment, that q_M is a good homotopical substitute for the inclusion map $\partial M \rightarrow M$.

The modest advantage that q_M has for us, compared to the inclusion $\partial M \rightarrow M$, is that the canonical action of the homeomorphism group $\text{homeo}(M)$ on the map q_M extends rather obviously to an action of the homeomorphism group $\text{homeo}(M \setminus \partial M)$ on q_M (because $M/\partial M$ is the one-point compactification of $M \setminus \partial M$.) Of course, $\text{homeo}(M)$ also acts canonically on the inclusion $\partial M \rightarrow M$, or equivalently on the pair $(M, \partial M)$, but this action does not extend obviously or otherwise to an action of $\text{homeo}(M \setminus \partial M)$ except in the cases where $\partial M = \emptyset$ or $\dim(M) \leq 1$.

We take the view that $\text{homeo}(M \setminus \partial M)$ is an enlargement of $\text{homeo}(M)$. Indeed, the restriction homomorphism from $\text{homeo}(M)$ to $\text{homeo}(M \setminus \partial M)$ is injective, due to the fact that $M \setminus \partial M$ is dense in M .

Let Z_M be the space of maps $w: [0, 1] \rightarrow M$ such that $w^{-1}(\partial M) = \{0\}$. Then we have a map $Z_M \rightarrow \text{holink}(M/\partial M, \star)$ given by composing elements $w \in Z_M$ with the quotient map $M \rightarrow M/\partial M$. This map is a weak homotopy equivalence. There is also a forgetful map $Z_M \rightarrow \partial M$ given by evaluation at 0. Together these maps make up a diagram

$$\begin{array}{ccc} \text{holink}(M/\partial M, \star) & \xrightarrow{q_M} & M \setminus \partial M \\ \simeq \uparrow & & \downarrow \simeq \\ Z_M & \xrightarrow{\simeq} & \partial M \xrightarrow{\text{incl.}} M \end{array}$$

which is commutative up to a preferred homotopy. The group $\text{homeo}(M)$ acts on the whole diagram, respecting the preferred homotopy. For the top row that action extends to an action of $\text{homeo}(M \setminus \partial M)$.

To package some of this in a more memorable way, let us write M_- for $M \setminus \partial M$ and $\partial^h M_-$ for $\text{holink}(M/\partial M, \star)$. Then we have

$$q_M: \partial^h M_- \longrightarrow M_-.$$

As we have seen, this map is a good homotopical substitute for the inclusion of ∂M in M . The group $\text{homeo}(M_-)$ acts on q_M . The action determines a map

$$B\text{homeo}(M_-) \longrightarrow B\text{haut}(q_M: \partial^h M_- \rightarrow M_-)$$

where $\text{haut}(\dots)$ generally denotes a space of derived homotopy automorphisms of an object in some model category. (A map such as q_M should be viewed as a functor from the totally ordered set $\{0, 1\}$ to spaces. We use one of the standard model category structures on the category of such functors.)

5.2. The prototype. Suppose that the compact smooth M satisfies the condition of [10, Thm. 2.1.1]. That is to say, M is the total space of a smooth disk bundle on a smooth compact manifold L without boundary, where the fibers are of dimension $c \geq 3$. Write

$$\text{haut}_{N\text{Fin}}(\text{con}(M \setminus \partial M))$$

for the union of the homotopy invertible (path) components in $\mathbb{R}\text{map}_{N\text{Fin}}(X, X)$ where $X = \text{con}(M \setminus \partial M)$. Composition makes this into a grouplike topological or simplicial monoid.

THEOREM 5.2.1. *Under these conditions on M , the broken arrow in the following homotopy commutative diagram can be supplied:*

$$\begin{array}{ccc} B\text{homeo}(M_-) & \xrightarrow{\text{action}} & Bhaut(\partial^h M \rightarrow M_-) \\ \parallel & & \uparrow \text{dotted} \\ B\text{homeo}(M_-) & \xrightarrow{\text{action}} & Bhaut_{N\text{Fin}}(\text{con}(M_-)) \end{array}$$

PROOF. Write X for $\text{con}(M_-)$; we use the particle model. Let $A = \text{con}(M_-; 0)$ be the simplicial subspace of X obtained by allowing only configurations of cardinality zero. This is of course rather trivial: A_r is a point for all $r \geq 0$. We will be interested in $\psi_X: E^\sigma(X) \rightarrow X$ and in the section space $\mathbb{R}\Gamma(\psi_X)$ and also in the section space $\mathbb{R}\Gamma(\psi_X|_A)$. It has already been indicated that $\mathbb{R}\Gamma(\psi_X)$ is weakly equivalent to ∂M , and it is easy to show directly that $\mathbb{R}\Gamma(\psi_X|_A)$ is weakly equivalent to M or to M_- . We need a more precise statement.

- (i) The action of $\text{homeo}(M_-)$ on $q_M: \partial^h M \rightarrow M_-$ gives rise to two fiber bundles on $B\text{homeo}(M_-)$, with fibers $\partial^h M$ and M_- respectively, and a map from the first to the second. To this we refer loosely as a *pair* of fiber bundles on $B\text{homeo}(M_-)$.
- (ii) The action of $\text{haut}_{N\text{Fin}}(X)$ on $\mathbb{R}\Gamma(\psi_X) \rightarrow \mathbb{R}\Gamma(\psi_X|_A)$ gives rise to two fibrations on $B\text{haut}_{N\text{Fin}}(X)$, with fibers $\mathbb{R}\Gamma(\psi_X)$ and $\mathbb{R}\Gamma(\psi_X|_A)$ respectively, and a map from the first to the second. To this we refer loosely as a *pair* of fibrations on $B\text{haut}_{N\text{Fin}}(X)$.
- (iii) Under pullback along the inclusion

$$B\text{homeo}(M_-) \longrightarrow Bhaut_{N\text{Fin}}(\text{con}(M_-)) = Bhaut_{N\text{Fin}}(X),$$

the fibration pair in (ii) becomes fiberwise homotopy equivalent to the fiber bundle pair in (i). More precisely, there is a zig-zag of fiberwise weak equivalences over $B\text{homeo}(M_-)$, etc.

What we have to prove is (iii). To keep notation manageable, we will concentrate on the boundary fibrations with fibers $\mathbb{R}\Gamma(\psi_X)$ and $\partial^h M$, neglecting the fibrations with fibers $\mathbb{R}\Gamma(\psi_X|_A)$ and M_- . The main ideas are already in place and we just arrange them by introducing a simplicial map denoted

$$\varphi_X: X^\# \longrightarrow X,$$

closely related to $\psi_X : E^\sigma(X) \rightarrow X$. For $r \geq 0$, the space X_r^\sharp is an open subspace of

$$X_r \times \partial^h M_-$$

consisting of all pairs (y, w) where $y \in X_r$ and $w : [0, 1] \rightarrow M/\partial M$ is an element of $\partial^h M_-$ such that $w(t)$ for $t > 0$ is not contained in the *support* of y . (The support of y is a compact subset of M_- . It is the union of all finite subsets which arise as images of the various maps from finite sets to M_- which make an appearance in the description of y .)

- (iv) The inclusion $X^\sharp \rightarrow X \times \partial^h M_-$ is a degreewise weak equivalence (where $\partial^h M_-$ should be viewed as a constant simplicial space).
- (v) There is a map $X^\sharp \rightarrow E^\sigma(X)$ over X , given by taking a pair $(y, w) \in X_r$ to the element of $E_r^\sigma(X)$ obtained by evaluating w at 1 and using that value to increase the cardinality of all configurations in y by one.
- (vi) That map $X^\sharp \rightarrow E^\sigma(X)$ over X induces a weak equivalence of derived section spaces,

$$\mathbb{R}\Gamma(\varphi_X : X^\sharp \rightarrow X) \longrightarrow \mathbb{R}\Gamma(\psi_X : E^\sigma(X) \rightarrow X)$$

by [10, Thm. 2.1.1], translated to the configuration category setting.

- (vii) Projection from X^\sharp to $\partial^h M_-$ determines a weak equivalence

$$\mathbb{R}\Gamma(\varphi_X : X^\sharp \rightarrow X) \longrightarrow \partial^h M_-.$$

The weak equivalences in (vi) and (vii) respect canonical actions of $\text{homeo}(M_-)$. Equivalently, they extend to fiberwise weak homotopy equivalences between fibrations over $B\text{homeo}(M_-)$. This gives us the zig-zag of fiberwise weak homotopy equivalences which we require. \square

5.3. Variant with cardinality restriction. For a more technical variant of theorem 5.2.1 we need the Postnikov decomposition of the map $\partial M \rightarrow M$. For an integer $a \geq 0$ there is the factorization

$$\partial M \longrightarrow \wp_a \partial M \longrightarrow M$$

of $\partial M \rightarrow M$ where $\wp_a \partial M$ is obtained from ∂M , as a space over M , by killing the relative homotopy groups of $\partial M \rightarrow M$ in dimensions $\geq a + 2$. So $\partial M \rightarrow \wp_a \partial M$ is $(a + 1)$ -connected. We apply this construction (and use the informal notation) mutatis mutandis with $\partial^h M_- \rightarrow M_-$ instead of $\partial M \rightarrow M$.

For M satisfying the condition of [10, Thm. 2.1.1] and an integer $j \geq 1$, let $\text{con}(M_-; j)$ be the truncated configuration category where only configurations of cardinality $\leq j$ are allowed. The condition on M is, imprecisely stated, that M is the total space of a smooth disk bundle of fiber dimension $c \geq 3$ on a smooth closed manifold. The truncated configuration category $\text{con}(M_-; j)$ is still a fiberwise complete Segal space over $N\text{Fin}$.

THEOREM 5.3.1. *Under these conditions on M , the broken arrow in the following homotopy commutative diagram can be supplied:*

$$\begin{array}{ccc} B\text{homeo}(M_-) & \xrightarrow{\text{action}} & B\text{haut}(\wp_{(j+1)(c-2)} \partial^h M_- \longrightarrow M_-) \\ \parallel & & \uparrow \cdots \\ B\text{homeo}(M_-) & \xrightarrow{\text{action}} & B\text{haut}_{N\text{Fin}}(\text{con}(M_-; j)) \end{array}$$

The proof of this follows the lines of the proof of theorem 5.2.1 but relies more on the connectivity estimates in [10, Thm. 2.1.1]. \square

5.4. Variant with gate. In the next two theorems some unsystematic notation is used to describe spaces of automorphisms of pairs and more complicated situations. Fix M , a smooth manifold with boundary and corners, so that $\partial M = \partial_0 M \cup \partial_1 M$ and $\partial \partial_0 M = \partial \partial_1 M = \partial_0 M \cap \partial_1 M$. We write, rather unsystematically,

$$M_- := M \setminus \partial_1 M, \quad \partial_0 M_- = \partial_0 M \setminus \partial \partial_0 M$$

and $\partial_1^h M_-$ for $\text{holink}(M/\partial_1 M, \star)$, as well as $\partial^h \partial_0 M_-$ for $\text{holink}(\partial_0 M/\partial \partial_0 M, \star)$. There is a commutative square

$$\begin{array}{ccc} \partial^h \partial_0 M_- & \longrightarrow & \partial_0 M_- \\ \downarrow \text{incl.} & & \downarrow \text{incl.} \\ \partial_1^h M_- & \longrightarrow & M_- \end{array}$$

where the horizontal maps are given by evaluation of paths at time 1. This is our preferred homotopical substitute for the square of inclusion maps

$$\begin{array}{ccc} \partial \partial_0 M & \longrightarrow & \partial_0 M \\ \downarrow & & \downarrow \\ \partial_1 M & \longrightarrow & M \end{array}$$

and the advantage of the substitute over the original is that the homeomorphism group $\text{homeo}(M_-; \partial_0 M_-)$ acts in a canonical way. (More unsystematic notation here: $\text{homeo}(M_-; \partial_0 M_-)$ consists of homeomorphisms $M_- \rightarrow M_-$ which restrict to the identity on the boundary $\partial_0 M_-$.)

One more abbreviation: let U be a standard open collar neighborhood of $\partial_0 M_-$ in M_- . We assume that the closure of U in M_- is a smooth closed collar. Write $U_- := U \cap M_-$. Although it is a little careless, where homeomorphisms $M_- \rightarrow M_-$ are mentioned which restrict to the identity on $\partial_0 M_-$, we may mean homeomorphisms $M_- \rightarrow M_-$ which restrict to the identity on all of U .

THEOREM 5.4.1. *If M satisfies the conditions of [10, Thm. 3.2.1], then the broken arrow in the following homotopy commutative diagram can be supplied:*

$$\begin{array}{ccc} B\text{homeo}(M_-; \partial_0 M_-) & \xrightarrow{\text{action}} & \text{haut} \left(\begin{array}{ccc} \partial^h \partial_0 M_- & \longrightarrow & \partial_0 M_- \\ \downarrow & & \downarrow \\ \partial_1^h M_- & \longrightarrow & M_- \end{array} ; \text{ top row} \right) \\ \parallel & & \uparrow \text{dotted} \\ B\text{homeo}(M_-; \partial_0 M_-) & \xrightarrow{\text{action}} & B\text{haut}_{N\text{Fin}_*}(\text{con}(M_-) ; \text{con}(U_-)) \end{array}$$

For clarification, this theorem has theorem 5.2.1 as a special case. It is the special case where $\partial_0 M$ is empty.

OUTLINE OF A PROOF. Although the proof is in many ways similar to the proof of theorem 5.2.1, it does require and use one additional idea. The outline will concentrate on that.

Let $X = \mathbf{con}(M_-)$ and $\partial X := \mathbf{con}(U_-)$, both to be viewed as a complete Segal spaces over $N\mathbf{Fin}_*$. As indicated earlier we can pretend that $\mathbf{homeo}(M_-; \partial_0 M_-)$ consists of the homeomorphisms $M_- \rightarrow M_-$ which restrict to the identity on the closure of the collar U . This is necessary to make the lower horizontal arrow in the square (of the theorem) meaningful.

We proceed initially as in the proof of theorem 5.2.1. In particular we have $A \subset X$ as before. We are guided by the idea that the restriction map

$$\mathbb{R}\Gamma(\psi_X) \rightarrow \mathbb{R}\Gamma(\psi_X|_A)$$

is weakly equivalent to the inclusion map $\partial_1 M \rightarrow M$, or to the homotopical substitute

$$\partial_1^h M_- \longrightarrow M_-.$$

(The evidence comes from [10, Thm. 3.2.1] and the partial reformulation at the very end of example 3.3.1.) But we are chiefly interested in homotopy automorphisms of the square

$$(5.4.1) \quad \begin{array}{ccc} \partial^h \partial_0 M_- & \longrightarrow & \partial_0 M_- \\ \downarrow & & \downarrow \\ \partial_1^h M & \longrightarrow & M_- \end{array}$$

fixing the top row pointwise. Therefore it seems that we need to come to terms with a semi-combinatorial analogue of diagram (5.4.1) in the shape of a square

$$(5.4.2) \quad \begin{array}{ccc} \partial^h \partial_0 M_- & \longrightarrow & \partial_0 M_- \\ \downarrow & & \downarrow \\ \mathbb{R}\Gamma(\psi_X) & \longrightarrow & \mathbb{R}\Gamma(\psi_X|_A). \end{array}$$

This is in agreement with the proof of theorem 5.2.1; there we had $\partial_0 M_- = \emptyset$. In general, we do *not* (yet) have a sufficiently well understood combinatorial expression for $\partial^h \partial_0 M_-$ in terms of the configuration categories X and/or ∂X . Therefore diagram (5.4.2) is the hybrid that it is. — The new and perhaps slightly unexpected task therefore is to set up diagram (5.4.2) in such a way that the actions of $\mathbf{aut}_{N\mathbf{Fin}_*}(X; \partial X)$ and $\mathbf{haut}_{N\mathbf{Fin}_*}(X; \partial X)$ on the lower row extend to actions on the entire square which are trivial on the terms in the top row. One solution is to construct diagram (5.4.2) as the contraction of a bigger commutative diagram

$$(5.4.3) \quad \begin{array}{ccc} \partial^h \partial_0 M_- & \longrightarrow & \partial_0 M_- \\ \downarrow a & & \downarrow \\ \mathbb{R}\Gamma(\theta_{\mathbf{tw}(X)}) & \longrightarrow & \mathbb{R}\Gamma(\theta_{\mathbf{tw}(X)}|_{\mathbf{tw}(A)}) \\ \downarrow & & \downarrow \\ \mathbb{R}\Gamma(\psi_X) & \longrightarrow & \mathbb{R}\Gamma(\psi_X|_A). \end{array}$$

where we use the notation of section 4.2. Here the lower square is combinatorial, i.e., expressed in homotopical terms of X and ∂X . Then $\mathbf{haut}_{N\mathbf{Fin}_*}(X; \partial X)$ can act on the lower square, trivially on the upper row (of the lower square). The commutative diagram in example 4.2.2 together with [10, Prop. 4.2.1] and lemma A.1 below make this possible. Those actions can then be extended canonically to actions on

the entire diagram (5.4.3) which are trivial on the terms in the top square. — From this point onwards, the proof is a straightforward adaptation of the proof of theorem 5.2.1. \square

The arrow labeled a in diagram (5.4.3) is not claimed to be a weak equivalence. The argument does not require that either, but [10, §4.2] has some suggestions that it is a weak equivalence under some additional geometric hypotheses on M .

5.5. Variant with gate and cardinality restriction. We keep the notation of section 5.4 and combine with the notation of section 5.3 for truncated configuration categories and Postnikov decompositions. Specifically (and unsystematically),

$$\wp_a \partial_1^h M_- \longrightarrow M_-$$

is the map obtained from $\partial_1^h M_- \longrightarrow M_-$ by killing the relative homotopy groups in dimensions $\geq a + 2$. There is still a commutative square

$$\begin{array}{ccc} \partial^h \partial_0 M_- & \longrightarrow & \partial_0 M_- \\ \downarrow & & \downarrow \\ \wp_{(j+1)(c-2)} \partial_1^h M_- & \longrightarrow & M_- \end{array}$$

THEOREM 5.5.1. *If M satisfies the conditions of [10, Thm. 3.2.1], then the broken arrow in the following homotopy commutative diagram can be supplied:*

$$\begin{array}{ccc} B\text{homeo}(M_-; \partial_0 M_-) & \xrightarrow{\text{action}} & \text{haut} \left(\begin{array}{ccc} \partial^h \partial_0 M_- & \longrightarrow & \partial_0 M_- \\ \downarrow & & \downarrow \\ \wp_{(j+1)(c-2)} \partial_1^h M_- & \longrightarrow & M_- \end{array} ; \text{top row} \right) \\ \parallel & & \uparrow \text{dotted} \\ B\text{homeo}(M_-; \partial_0 M_-) & \xrightarrow{\text{action}} & B\text{haut}_{N\text{Fin}_*}(\text{con}(M_-; j) ; \text{con}(U_-; j)) \end{array}$$

This theorem has theorem 5.3.1 as a special case, the case where $\partial_0 M = \emptyset$ and consequently $\partial_1 M = \partial M$.

A. Locality property of holim Θ

We use the notation of [10, §4]. So M is a smooth compact manifold with boundary and corners in the boundary, $\partial M = \partial_0 M \cup \partial_1 M$ etc., and \mathcal{P} is short for $\mathcal{P}(M \setminus \partial_1 M)$. For an object (S, ρ) of \mathcal{P} let $V_{\text{col}}(S, \rho)$ be the collar part of the open subset $V(S, \rho)$ in $M \setminus \partial_1 M$.

A topological poset \mathcal{Q} is defined [10, §4.2] as a quotient of $\text{tw}(\mathcal{P})$. Two elements of $\text{tw}(\mathcal{P})$, say $((S, \rho) \leq (T, \sigma))$ and $((S', \rho') \leq (T', \sigma'))$, determine the same element of \mathcal{Q} if and only if $V_{\text{col}}(T, \sigma) = V_{\text{col}}(T', \sigma')$ and

$$V_{\text{col}}(T, \sigma) \cap V(S, \rho) = V_{\text{col}}(T', \sigma') \cap V(S', \rho').$$

Therefore every element of \mathcal{Q} has a unique representative in $\text{tw}(\mathcal{P})$ of the form $((S, \rho) \leq (\emptyset, \sigma))$. For elements of \mathcal{Q} represented in this way by $((S, \rho) \leq (\emptyset, \sigma))$ and $((S', \rho') \leq (\emptyset, \sigma'))$, respectively, the first is \leq the second in \mathcal{Q} if and only if $V(\emptyset, \sigma') \subset V(\emptyset, \sigma)$ and $V(S, \rho) \cap V(\emptyset, \sigma') \subset V(S', \rho')$.

The name of the quotient functor from $\text{tw}(\mathcal{P})$ to \mathcal{Q} is K . It is clear that $\Theta = \Theta_1 \circ K$ for a unique Θ_1 from \mathcal{Q} to spaces.

LEMMA A.1. *The map $\mathrm{holim} \Theta_1 \longrightarrow \mathrm{holim} \Theta_1 \circ K = \mathrm{holim} \Theta$ induced by the forgetful functor $K: \mathrm{tw}(\mathcal{P}) \rightarrow \mathcal{Q}$ is a weak equivalence.*

PROOF. There is a routine reduction (which is skipped here) to the discrete setting; see [10, §1.2]. Let $\delta\mathcal{P}$ and $\delta\mathcal{Q}$ be the discrete posets obtained from \mathcal{P} and \mathcal{Q} , respectively. Now we need to show that the canonical map

$$\mathrm{holim} \Theta_1|_{\delta\mathcal{Q}} \longrightarrow \mathrm{holim} \Theta_1 K|_{\mathrm{tw}(\delta\mathcal{P})}$$

is a weak equivalence. We start by introducing a topological poset \mathcal{U} intermediate between $\mathrm{tw}(\delta\mathcal{P})$ and $\delta\mathcal{Q}$. This is also a quotient of $\mathrm{tw}(\delta\mathcal{P})$. Two elements of $\mathrm{tw}(\delta\mathcal{P})$, say $((S, \rho) \leq (T, \sigma))$ and $((S', \rho') \leq (T', \sigma'))$, determine the same element of \mathcal{U} if and only if $V_{\mathrm{col}}(T, \sigma) = V_{\mathrm{col}}(T', \sigma')$ and $(S, \rho) = (S', \rho')$. For elements of \mathcal{U} represented in this way by $((S, \rho) \leq (T, \sigma))$ and $((S', \rho') \leq (T', \sigma'))$ in $\mathrm{tw}(\delta\mathcal{P})$, the first is \leq the second if and only if

- $V(S, \rho) \subset V(S', \rho')$;
- $V_{\mathrm{col}}(T', \sigma') \subset V_{\mathrm{col}}(T, \sigma)$;
- $V(S', \rho')$ is in *general position* to $V_{\mathrm{col}}(T, \sigma)$. This means that for every connected component W of $V(S', \rho')$, either $W \subset V_{\mathrm{col}}(T, \sigma)$ or the closure of W has empty intersection with the closure of $V_{\mathrm{col}}(T, \sigma)$.

Now the forgetful functor K (in the discrete setting) is the composition of two forgetful functors K_s and K_t :

$$\mathrm{tw}(\delta\mathcal{P}) \xrightarrow{K_t} \mathcal{U} \xrightarrow{K_s} \delta\mathcal{Q}.$$

The plan is to show that both induced maps

$$\mathrm{holim} \Theta_1^\delta \longrightarrow \mathrm{holim} \Theta_1^\delta K_s \longrightarrow \mathrm{holim} \Theta_1^\delta K_s K_t$$

are weak equivalences (where Θ_1^δ is Θ_1 restricted to $\delta\mathcal{Q}$). It suffices to establish the property *homotopy terminal* for the forgetful functors $K_t: \mathrm{tw}(\delta\mathcal{P}) \rightarrow \mathcal{U}$ and $K_s: \mathcal{U} \rightarrow \delta\mathcal{Q}$.

Showing that K_t is homotopy terminal. Fix an object z of \mathcal{U} . We can represent this by an object of $\mathrm{tw}(\delta\mathcal{P})$, say

$$((S^\sharp, \rho^\sharp) \leq (T^\sharp, \sigma^\sharp)).$$

The category $(z \downarrow K_t)$ is identified with a full sub-poset \mathcal{A} of $\mathrm{tw}(\delta\mathcal{P})$, consisting of all objects

$$((S, \rho) \leq (T, \sigma)) \in \mathrm{tw}(\delta\mathcal{P})$$

such that $(S^\sharp, \rho^\sharp) \leq (S, \rho)$ in $\delta\mathcal{P}$ and $V_{\mathrm{col}}(T, \sigma) \subset V(T^\sharp, \sigma^\sharp)$, and $V(S, \rho)$ is in general position to $V_{\mathrm{col}}(T^\sharp, \sigma^\sharp)$. (See the earlier description of \mathcal{U} .) The poset \mathcal{A} has a full sub-poset \mathcal{B} consisting of all $((S, \rho) \leq (T, \sigma)) \in \mathcal{A}$ where

$$(S, \rho) = (S^\sharp, \rho^\sharp).$$

For every $y \in \mathcal{A}$, the full sub-poset of \mathcal{B} consisting of the elements of \mathcal{B} which are $\leq y$ in \mathcal{A} has a unique maximum. Indeed, if $y = ((S, \rho) \leq (T, \sigma))$ as above then that maximum is $((S^\sharp, \rho^\sharp) \leq (T, \sigma))$. Equivalently, the inclusion $\mathcal{B} \rightarrow \mathcal{A}$ has a right adjoint. Now \mathcal{B} has a maximal element given by $((S^\sharp, \rho^\sharp) \leq (S^\sharp, \rho^\sharp))$. Therefore the classifying space of \mathcal{B} is contractible.

Showing that K_s is homotopy terminal. Fix an object x of $\delta\mathcal{Q}$ represented by $((S^\flat, \rho^\flat) \leq (T^\flat, \sigma^\flat))$ in $\mathrm{tw}(\delta\mathcal{P})$ in such a way that $T^\flat = \emptyset$. We need to show that

the poset $(x \downarrow K_s)$ has a contractible classifying space. Identify $(x \downarrow K_s)$ with the full sub-poset \mathcal{D} of \mathcal{U} consisting of all objects of \mathcal{U} represented by

$$((S, \rho) \leq (T, \sigma)) \in \text{tw}(\delta\mathcal{P})$$

which satisfy

- (i) $V_{\text{col}}(S^b, \rho^b) \subset V_{\text{col}}(T, \sigma) \subset V(T^b, \sigma^b)$;
- (ii) $V(S, \rho) \cap V_{\text{col}}(T, \sigma) \supset V(S^b, \rho^b) \cap V_{\text{col}}(T, \sigma)$.

Let \mathcal{E} be the full sub-poset of \mathcal{D} consisting of all objects as above which instead of (ii) satisfy the stronger condition

- (iii) $V(S, \rho) \cap V_{\text{col}}(T, \sigma) = V(S^b, \rho^b) \cap V_{\text{col}}(T, \sigma)$.

For every $y \in \mathcal{D}$ the full poset of the elements of \mathcal{E} which are $\leq y$ in \mathcal{D} has a unique maximum. Equivalently, the inclusion $\mathcal{E} \rightarrow \mathcal{D}$ has a right adjoint. It remains to show that \mathcal{E} has a contractible classifying space. We show this in a separate step.

Showing that \mathcal{E} has a contractible classifying space. Let \mathcal{J} be the full sub-poset of $\delta\mathcal{P}$ consisting of the $(T, \sigma) \in \delta\mathcal{P}$ which have $T = \emptyset$ and satisfy the following additional conditions:

- $V_{\text{col}}(S^b, \rho^b) \subset V(T, \sigma) \subset V_{\text{col}}(T^b, \sigma^b)$;
- $V(T, \sigma) = V_{\text{col}}(T, \sigma)$ is in general position to $V(S^b, \rho^b)$.

Let G be the functor from \mathcal{J}^{op} to posets which

- to an object $(T, \sigma) \in \mathcal{J}$ associates the full sub-poset of \mathcal{P} consisting of all $(S, \rho) \in \mathcal{P}$ such that $V_{\text{col}}(S, \rho) = V(T, \sigma)$;
- to a morphism $(T, \sigma) \leq (T', \sigma')$ in \mathcal{J} associates the map of posets

$$G(T', \sigma') \ni (S', \rho') \mapsto (S, \rho) \in G(T, \sigma)$$

where (S, ρ) is defined in such a way that $V_{\text{col}}(S, \rho) = V(T, \sigma)$ and

$$V(S, \rho) \setminus V_{\text{col}}(S, \rho)$$

is the (disjoint) union of $V(S', \rho') \setminus V_{\text{col}}(S', \rho')$ and the part of $V(S^b, \rho^b)$ contained in $V(T', \sigma') \setminus V(T, \sigma)$.

Each $G(T, \sigma)$ has a contractible classifying space; indeed it has a minimal element. It is also easy to see that \mathcal{J}^{op} has a contractible classifying space. Therefore the classifying space of the Grothendieck construction $\int G$ is contractible, e.g. by the Thomason homotopy colimit theorem [9]. But $\int G$ is clearly equivalent to \mathcal{E}^{op} . (By the Grothendieck construction $\int F$ of a functor F from a small category \mathcal{M} to the category of small categories we mean the following category. Objects are pairs (m, v) where m is an object of \mathcal{M} and v is an object of $F(m)$. A morphism from (m, v) to (n, w) is a pair (f, g) where $f: m \rightarrow n$ is a morphism in \mathcal{M} and $g: F(f)(v) \rightarrow w$ is a morphism in $F(n)$.) \square

By combining lemma A.1 with the locality result of [10, §4.2] for $\text{holim } \Theta_1$, we obtain a similar locality result for $\text{holim } \Theta$.

References

1. Ricardo Andrade, Ph.D thesis, MIT 2010.
2. J. F. Adams, *Infinite loop spaces*, Annals of Math. Studies 90, Princeton Univ. Press 1978.
3. P. Boavida de Brito and M.S. Weiss, *Spaces of smooth embeddings and configuration categories*, preprint, arXiv:1502.01640
4. W. G. Dwyer and D. Kan, *Function complexes in homotopical algebra*, Topology 19 (1980), 427–440.

5. P. S. Hirschhorn, *Model categories and their localizations*, Math. Surveys and Monographs vol. 99, Amer.Math.Soc., 2002.
6. M. Hovey, *Model categories*, Mathematical Surveys and Monographs, 63. Amer.Math.Soc., Providence, RI, 1999. xii+209 pp.
7. C. Rezk, *A model for the homotopy theory of homotopy theory*, Trans.Amer.Math.Soc. 353 (2001), 973-1007.
8. G. Segal, *Categories and cohomology theories*, Topology 13 (1974), 293-312.
9. R.W. Thomason, *Homotopy colimits in the category of small categories*, Math. Proc. Cambridge Philos. Soc. 85 (1979), 91-109
10. S. Tillmann and M. S. Weiss, *Occupants in manifolds*, arXiv:1503.00498
11. M. Weiss, *Dalian notes on Pontryagin classes*, arXiv:1507.00153

MATH. INSTITUT, UNIVERSITÄT MÜNSTER, 48149 MÜNSTER, EINSTEINSTRASSE 62, GERMANY
E-mail address: `m.weiss@uni-muenster.de`